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GENERALIZED SCALAR CURVATURES OF COHOMOLOGICAL EINSTEIN KAEHLER MANIFOLDS

KOICHI OGIUE

1. Introduction

In Riemannian geometry all elementary symmetric polynomials of eigenvalues of the Ricci tensor are geometric invariants. In particular, the one of degree 1 is called the scalar curvature.

In this paper, we shall study some properties of the geometric invariants for *cohomological Einstein* Kaehler manifolds. Let M be a Kaehler manifold with fundamental 2-form Φ and Ricci 2-form γ. We say that M is cohomologically Einsteinian if $[\gamma] = a \cdot [\Phi]$ for some constant a, where [*] denotes the cohomology class represented by *. It is well-known that the first Chern class $c_1(M)$ is represented by γ.

Let $z_1, \cdots, z_n$ be a local coordinate system in M, $g = \sum g_{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta$ be the Kaehler metric of M, and $S = \sum R_{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta$ be the Ricci tensor of M. Define n scalars $\rho_1, \cdots, \rho_n$ by

$$\frac{\det(g_{\alpha\bar{\beta}} + tR_{\alpha\bar{\beta}})}{\det(g_{\alpha\bar{\beta}})} = 1 + \sum_{k=1}^{n} \rho_k t^k ,$$

and denote the scalar curvature of M by ρ. Then it is easily seen that $\rho = 2\rho_1$, and is also clear that $\rho_n = \det(R_{\alpha\bar{\beta}})/\det(g_{\alpha\bar{\beta}})$.

We shall prove

Theorem 1. *Let M be an n-dimensional compact cohomological Einstein Kaehler manifold. If $c_1(M) = a \cdot [\Phi]$, then*

$$\int_M \rho_k * 1 = (2\pi a)^k \binom{n}{k} \int_M * 1 ,$$

where $\binom{n}{k}$ denotes the binomial coefficient, and $*1$ the volume element of M.

This results implies that the average of ρ_k, $\int_M \rho_k * 1 \Big/ \int_M *1$, does not depend on the metric too strongly.

Let $P_{n+p}(C)$ be an $(n + p)$-dimensional complex projective space with the

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Fubini-Study metric of constant holomorphic sectional curvature 1. An n-dimensional algebraic manifold imbedded in $P_{n+p}(C)$ is called a *complete intersection manifold* if M is given as an intersection of p nonsingular hypersurfaces $M_1, \cdots, M_p$ in $P_{n+p}(C)$, i.e., if $M = M_1 \cap \cdots \cap M_p$. It is known that the (first) Chern class of a complete intersection manifold M is completely determined by the degrees of $M_1, \cdots, M_p$, and it is easily seen that a complete intersection manifold is cohomologically Einsteinian with respect to the induced Kaehler metric.

Theorem 2. *Let M be an n-dimensional complete intersection manifold in $P_{n+p}(C)$, i.e., let $M = M_1 \cap \cdots \cap M_p$. Then*

$$\int_M \rho_k * 1 = \binom{n}{k}\left[\frac{1}{2}(n + p + 1 - \sum a_\alpha)\right]^k (\prod a_\alpha)\frac{(4\pi)^n}{n!} ,$$

where a_α denotes the degree of M_α, $\alpha = 1, \cdots, p$.

Theorem 3. *Let M be an n-dimensional complete intersection manifold in $P_{n+p}(C)$. If $\rho_k > \binom{n}{k}\left(\frac{n}{2}\right)^k$ for some k, then M is a linear subspace.*

The above theorems can be considered as generalizations of the results in [3]. Theorem 2 is of Gauss-Bonnet type in the sense that it provides a relationship between differential geometric invariants and more primitive invariants: The scalar ρ_k is a differential geometric invariant and depends fully on the equations defining M, but Theorem 2 implies that the integral of ρ_k depends only on (the sum and the product of) the degrees of M. Theorem 3 gives a characterization of a linear subspace among complete intersection manifolds.

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2. Proof of Theorem 1

Let Φ be the fundamental 2-form of M, that is, a closed 2-form defined by

$$\Phi = \frac{\sqrt{-1}}{2} \sum g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta . \tag{1}$$

Let γ be the Ricci 2-form of M, that is, a closed 2-form defined by

$$\gamma = \frac{\sqrt{-1}}{4\pi} \sum R_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta . \tag{2}$$

Then the first Chern class $c_1(M)$ is represented by γ. We denote [*] to be the cohomology class represented by a closed form * so that, in particular, $c_1(M) = [\gamma]$.

Since $c_1(M) = a \cdot [\Phi]$, there exists a 1-form η satisfying

$$\gamma = a\Phi + d\eta .$$

Therefore we obtain

$$(3)\qquad \gamma^k = a^k\Phi^k + \sum_{\ell=1}^{k} (\cdots)\, \Phi^{k-\ell} \wedge (d\eta)^\ell ,$$

where $(\cdots)$ is a constant involving ℓ.

Let Λ be the operator of interior product by Φ. Then it follows from (1) and (2) that

$$\Lambda^k\Phi^k = \frac{k!\,n!}{(n-k)!} , \qquad \Lambda^k\gamma^k = \frac{k!\,k!}{(2\pi)^k}\rho_k .$$

These, together with (3), imply

$$\frac{k!\,k!}{(2\pi)^k}\rho_k = a^k\frac{k!\,n!}{(n-k)!} + \sum_{\ell=1}^{k} (\cdots)\Lambda^k\Phi^{k-\ell} \wedge (d\eta)^\ell ,$$

so that

$$(4)\qquad \rho_k = (2\pi a)^k\binom{n}{k} + \sum_{\ell=1}^{k} \{\cdots\}\, \Lambda^\ell(d\eta)^\ell ,$$

where $\{\cdots\}$ is a constant involving ℓ.

Let δ be the codifferential operator, and C the operator defined by $C\alpha = (\sqrt{-1})^{r-s}\alpha$, where α is a form of bidegree (r, s). Then $\delta\Lambda = \Lambda\delta$, $C\Lambda = \Lambda C$ and $d\Lambda - \Lambda d = C^{-1}\delta C$ (cf. for example [1]). We can prove inductively that $d\Lambda^\ell - \Lambda^\ell d = \ell C^{-1}\delta C\Lambda^{\ell-1}$, from which it follows that $\Lambda^\ell(d\eta)^\ell = \Lambda^\ell d(\eta \wedge (d\eta)^{\ell-1})$ $= -\ell C^{-1}\delta C\Lambda^{\ell-1}(\eta \wedge (d\eta)^{\ell-1})$, and hence $\int_M \Lambda^\ell(d\eta)^\ell * 1 = 0$. Therefore from (4) we have

$$\int_M \rho_k * 1 = (2\pi a)^k\binom{n}{k}\int_M * 1 .$$

3. Proof of Theorems 2 and 3

Let $\tilde{h}$ be the generator of $H^2(P_{n+p}(C), Z)$ corresponding to the divisor class of a hyperplane in $P_{n+p}(C)$. Then the first Chern class $c_1(P_{n+p}(C))$ of $P_{n+p}(C)$ is given by

$$(5)\qquad c_1(P_{n+p}(C)) = (n + p + 1)\tilde{h} .$$

Let $j: M \to P_{n+p}(C)$ be the imbedding, and h the image of $\tilde{h}$ under the homomorphism $j^*: H^2(P_{n+p}(C), Z) \to H^2(M, Z)$. Then the first Chern class $c_1(M)$ of M is given by

$$(6)\qquad c_1(M) = (n + p + 1 - \sum a_\alpha)h \;.$$

Let $\tilde{\Phi}$ be the fundamental 2-form of $P_{n+p}(C)$. Since the Fubini-Study metric $\tilde{g}$ and the Ricci tensor $\tilde{S}$ of $P_{n+p}(C)$ are related by

$$\tilde{S} = \tfrac{1}{2}(n + p + 1)\tilde{g} \;,$$

the Ricci 2-form $\tilde{\gamma}$ of $P_{n+p}(C)$ satisfies

$$\tilde{\gamma} = \frac{n + p + 1}{4\pi}\tilde{\Phi} \;.$$

Therefore we have

$$(7)\qquad c_1(P_{n+p}(C)) = \frac{n + p + 1}{4\pi}[\tilde{\Phi}] \;.$$

Since $\Phi = j^*\tilde{\Phi}$, it follows from (5), (6) and (7) that

$$c_1(M) = \frac{n + p + 1 - \sum a_\alpha}{4\pi}[\Phi] \;,$$

which implies that M is cohomologically Einsteinian. Therefore from Theorem 1 we have

$$(8)\qquad \int_M \rho_k * 1 = \left[\frac{1}{2}(n + p + 1 - \sum a_\alpha)\right]^k \binom{n}{k} \int_M * 1 \;.$$

Let $P_p(C)$ be a p-dimensional linear subspace of $P_{n+p}(C)$, and ν the number of points in $M \cap P_p(C)$. Then the dimension theory for algeraic manifolds states that ν does not depend on the choice of $P_p(C)$ if $P_p(C)$ is in general position. By a theorem of Wirtinger [4], the volume of M is given by

$$\int_M * 1 = \nu\frac{(4\pi)^n}{n!} \;.$$

On the other hand, since M is a complete intersection maifold, we have [2]

$$\nu = \prod a_\alpha \;.$$

Therefore it follows that

$$\int_M * 1 = (\prod a_\alpha)\frac{(4\pi)^n}{n!} \;,$$

which, combined with (8), completes the proof of Theorem 2.

If $\rho_k > \binom{n}{k}\left(\frac{n}{2}\right)^k$, then it follows from (8) that

$$\binom{n}{k}\left(\frac{n}{2}\right)^k \int_M *1 < \left[\frac{1}{2}(n + p + 1 - \sum a_\alpha)\right]^k \binom{n}{k} \int_M *1 ,$$

which implies $\sum a_\alpha < p + 1$, that is, $a_1 = \cdots = a_p = 1$. This proves Theorem 3.

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MICHIGAN STATE UNIVERSITY